Effects of random noise on a simple class of growing network models

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We investigate the effects of random noise on network systems. In particular, we consider a simple class of growing network models whose topological structure is determined by the preferred attachment A_k. We introduce a noise-induced attachment $\tilde{A}_k$ which includes fluctuations in the number of links of individual nodes due to a random noise. We carry out the numerical simulations to show that the topological structure of the networks is determined not only by A_k but also by the strength of the noise. Analytic and numerical solutions are also presented to support this observation. In addition, we study the stability of networks against attacks under the noisy condition. Similarly, we introduce a noise-induced preferred deletion $\tilde{B}_k$, and show that noise is an essential feature to determine the stability of networks.

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I. INTRODUCTION

The properties of complex networks have been investigated by many scientists recently (for reviews, see Refs. [1–4]). In particular, Barabási and Albert [5] proposed a simple growing network model to explain a power-law distribution of nodes that have k links $n_k \sim k^{-\nu}$, which is often observed in many growing network systems including the World Wide Web, scientific collaboration, citation networks, etc. In their model, a new node links to an existing node with a probability that is proportional to the number of links of the existing node (the preferred attachment A_k). As a result they found a power-law distribution of nodes with a exponent $\nu = 3$, which roughly matches with the observed values of many real networks [4]. More general growing network models have also been proposed, either suggesting a generalized form of the attachment [6] or featuring new essential effects in network models [7].

In this paper, we investigate a simple class of growing network models under the influence of random noise. As far as we know, the effects of noise have not been considered in the previous network models because noise often seems to is relatively small and is averaged to zero in the statistical limit. However, it is questionable that if the above assumptions are applicable in real network systems. In particular, noise may influence us to miscount the exact number of links of individual nodes, which results in connecting a new node to an improper old node. The strength of noise is generally small. So, the effects of the noise may not be important for highly linked nodes. However, for nodes with only a few links, the effects of the noise cannot be ignored.

In Sec. II, we attempt to construct a noise-induced growing network model featuring the above consideration. We present the results of our numerical simulations in the following section. In Sec. III, we obtain analytical solutions of our model using the rate equation approach [6] and compare them with our simulations as well. In addition, we also show the effects of noise on the stability of networks in Sec. IV. Finally, we add a brief conclusion in Sec. V.

II. NOISE-INDUCED MODEL

We attempt to construct a simple class of growing network models that have an intrinsic random noise. We assume that a new node likely links to an old node with a probability that is proportional to the number of links of the target node (a preferred attachment A_k). That is, all old nodes with the same number of nodes have the same strength of attracting a new node. However, if noise exists in the network, nodes have different attachment probability even though they have the same number of links. Thus, we assume the individual attachment under the noisy condition as $\tilde{A}_k^{(i)} = A_k + r_k^{(i)}$ [where (i) denotes an individual node that has k links and $r_k^{(i)}$ is a random noise]. To most simply illustrate the effect of a random noise, we further assume $A_k = k^\alpha$ (where α is a small real number [8]) and the noise causes miscounting of the number of links of target nodes only. Then, the noise-induced attachment of node (i) is obtained as

$$\tilde{A}_k^{(i)} = \begin{cases} (\max[0, k + r^{(i)}])^\alpha & (\alpha > 0) \\ \text{const} & (\alpha = 0) \\ (\max[1, k + r^{(i)}])^\alpha & (\alpha < 0), \end{cases} \tag{1}$$

where $r^{(i)}$ is an uniformly distributed random integer ($r^{(i)} = -\Delta, -\Delta + 1, \ldots, \Delta$ and Δ is a positive integer) and independent of the number of links of a target node. (Note that by definition, $0 \leqslant \tilde{A}_k^{(i)} < \infty$.)

Incorporating this noise-induced attachment of individual node $\tilde{A}_k^{(i)}$, we carry out numerical simulations as follows: (i) we create a new node at each time step; (ii) for all existing nodes, we assign a random integer $r^{(i)}$ $(-\Delta \leqslant r^{(i)} \leqslant \Delta)$ and calculate $\tilde{A}_k^{(i)}$ from Eq. (1); (iii) we obtain a normalized preferred attachment probability of an old node (i) by dividing $\tilde{A}_k^{(i)}$ by the sum $\Sigma_{i=1}^N \tilde{A}_k^{(i)}$, where N is the total number of existing nodes; (iv) we link the new node to an old node selected by the attachment probability obtained in step (iii); (v) we repeat steps (i)–(iv) adding 10^5 nodes and calculate the probability distribution of nodes with k links, n_k; (vi) in order to obtain better statistics, we carry out ten simulations and finally calculate an ensemble average probability $\langle n_k \rangle$.

Figure 1 shows $\langle n_k \rangle$ versus k with different strengths of random noises. For the linear attachment ($\alpha = 1.0$), the slope of $\langle n_k \rangle$ becomes steeper and steeper in large k region as Δ

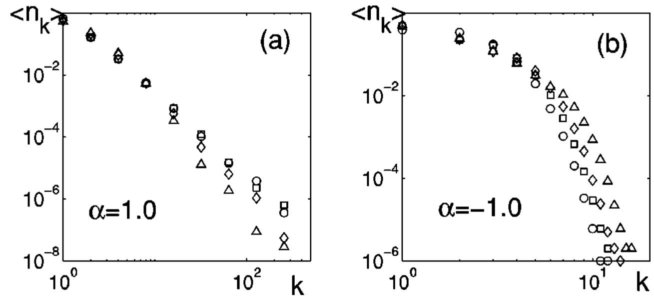

FIG. 1. $\langle n_k \rangle$ versus k using $\tilde{A}$, in Eq. (1). (a) For the linear attachment $\alpha=1.0$ with $\Delta=0$ (○), 1 (□), 5 (◇), and 10 (△), while (b) for the inverse linear attachment $\alpha=-1.0$ with $\Delta=0$ (○), 1 (□), 2 (◇), and 5(△). $\langle n_k \rangle$ is the average number of nodes obtained from ten numerical experiments for each parameter value. The total number of nodes in the networks is 10^5. [Note that data in the left plot are chosen when $k=2^n$ ($n=0,1,\ldots$).]

increases [Fig. 1(a)]. However, for the inverse linear attachment ($\alpha=-1.0$), the probability of having nodes with a small k decreases as Δ increases, which is the opposite to the previous observation [Fig. 1(b)].

III. ANALYSIS OF THE MODEL

If there is no noise, our model can be described by the rate equations for $N_k(t)$, the number of nodes that have k links,

$$\frac{dN_k}{dt}=\frac{A_{k-1}N_{k-1}-A_kN_k}{\sum_{j=1}^{\infty} A_jN_j}+\delta_{k,1}, \tag{2}$$

where the preferred attachment A_k is an *average* strength of old nodes with k links to attract a new node, which is calculated from the relation $A_k=\Sigma_{i=1}^{N_k}A_k^{(i)}/N_k$ [$A_k^{(i)}$ is attraction strength of the node (i) that has k links], and $\Sigma_{j=1}^{\infty}A_jN_j$ is the normalization factor. For a specific type of attachment $A_k \sim k^{\gamma}$ (where $0\leqslant\gamma<2$), analytic and asymptotic solutions of Eq. (2) have been studied by Krapivsky and Redner [6]. In particular, when $k\leqslant 1$, assuming that $N_k=n_kt$ and $\Sigma_{j=1}^{\infty}A_jN_j=\mu t$, they presented formal solutions of Eq. (2),

$$n_k=\frac{\mu}{A_k}\prod_{j=1}^{k}\left(1+\frac{\mu}{A_j}\right)^{-1}. \tag{3}$$

Furthermore, inserting Eq. (3) into the definition of $\mu=\Sigma_{j=1}^{\infty}A_jn_j$, μ is obtained by solving the following relation:

$$\sum_{k=1}^{\infty}\prod_{j=1}^{k}\left(1+\frac{\mu}{A_j}\right)^{-1}=1. \tag{4}$$

Therefore, if we know A_k, then we can calculate μ from Eq. (4) and with this μ we eventually obtain n_k. This result suggests that the topological structure of a network is determined by A_k. That is, it is crucial to measure A_k in order to investigate the real network systems.

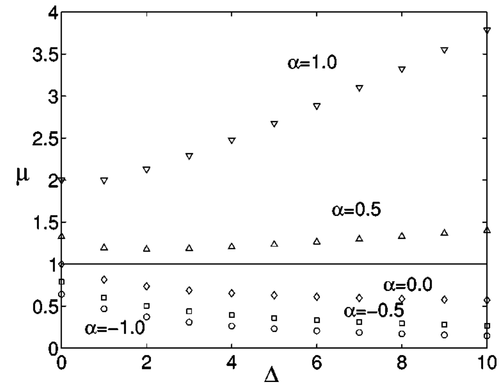

FIG. 2. μ versus Δ with $\alpha=1.0$ (▽), 0.5 (△), 0.0 (◇), -0.5 (□), and -1.0 (○). (Note that μ decreases as Δ increases when $\alpha=0.0$.)

Now, we introduce a random noise in the model. The individual attachment can be obtained from Eq. (1), and using this the appropriated attachment $\tilde{A}_k$ is obtained from the relation

$$\tilde{A}_k=\frac{\sum_{i=1}^{N_k}(k+r^{(i)})_{>0}^{\alpha}}{N_k}, \tag{5}$$

where $r^{(i)}$ is an intrinsic noise (which is an uniformly distributed random integer number in the region $-\Delta\leqslant r^{(i)}\leqslant\Delta$) and $\Sigma(\cdots)_{>0}^{\alpha}$ denotes the sum of the numerator only when $(\cdots)>0$ because we assume that all nodes have at least one link in our model. For large k ($k>\Delta$), applying the Taylor series expansion, Eq. (5) becomes $\tilde{A}_k\sim(k^{\alpha}/N_k)[N_k+(\alpha/k)\Sigma_{i=1}^{N_k}r^{(i)}]$. Since $\Sigma_{i=1}^{N_k}r^{(i)}\sim 0$ (in the statistical limit $N_k\gg 1$), we finally obtain $\tilde{A}_k\sim k^{\alpha}=A_k$. For small k ($k\leqslant\Delta$), we need more rigorous calculation due to the restriction on the sum of the numerator of Eq. (5). However, we generally find that $\tilde{A}_k\geqslant A_k$ ($\tilde{A}_k\leqslant A_k$) when $\alpha>0$ ($\alpha\leqslant 0$). Substituting this noise-induced attachment $\tilde{A}_k$ into Eqs. (3) and (4), we can obtain formal solutions for our noise-induced model. (See solid lines in Fig. 3.)

A plot of μ versus Δ for different α is shown in Fig. 2. μ generally increases (decreases) as Δ increases when $\alpha>0$ ($\alpha\leqslant 0$). The case $\alpha=0$ seems to be contradictory because we can obtain $\mu=1.0$ from Eq. (4) if the attachment is a constant. However, if noise exists, some nodes seem to have zero or a negative number of links. Since a new node cannot connect to these *isolated* nodes, $\tilde{A}_k$ is not a constant but becomes a function that depends on Δ and k. Therefore, even though $\alpha=0$, μ is also a function of Δ as shown in Fig. 2.

As examples, we consider two simple cases with integer random noises ($r^{(i)}=-\Delta,-\Delta+1,\ldots,\Delta$) in the following.

(i) For a linear attachment ($\alpha=1$), we obtain $\tilde{A}_k=[(\Delta+k)(\Delta+1+k)]/2(2\Delta+1)$ ($k\leqslant\Delta$) and $\tilde{A}_k=k$ ($k>\Delta$) from Eq. (5). Inserting $\tilde{A}_k$ instead of A_k in Eq. (4), we numerically calculate $\mu=3.78$ when $\Delta=10$ (see Fig. 2). Using this μ and $\tilde{A}_k$ (instead of A_k), we also obtain n_k from Eq. (3). In par-

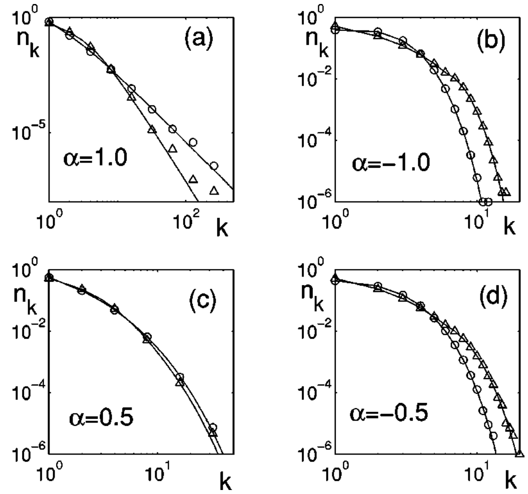

FIG. 3. n_k versus k. Solid lines represent numerical solutions obtained from Eqs. (5), (4), and (3), while the circles and the triangles are obtained by simulations (see Sec. II). (a) $\alpha=1.0$, $\Delta=0$ (○) and 10 (△), (b) $\alpha=-1.0$, $\Delta=0$ (○) and 5 (△), (c) $\alpha=0.5$, $\Delta=0$ (○) and 10 (△), and (d) $\alpha=-0.5$, $\Delta=0$ (○) and 5 (△). [Note that data in (a) and (c) are chosen when $k=2^n$ ($n=0,1,\ldots$).]

ticular, for sufficiently large k, $n_k \sim k^{-(1+\mu)} \sim k^{-4.78}$. Thus, the exponent of the node distribution is much larger than 3, which is obtained from the network without noise.

(ii) For an inverse linear attachment ($\alpha=-1$), we obtain $\widetilde{A}_k=[1/(2\Delta+1)]\Sigma_{i=1}^{k+\Delta}(1/i)$ $(k\leqslant\Delta)$ and $\widetilde{A}_k=[1/(2\Delta+1)]\Sigma_{i=k-\Delta}^{k+\Delta}(1/i)$ $(k>\Delta)$ from Eq. (5). Similarly, we calculate $\mu=0.23$ when $\Delta=5$ (see Fig. 2) and numerically obtain n_k from Eq. (3). For sufficiently large k, $\widetilde{A}_k$ becomes $1/k$ and n_k asymptotically becomes $(k\mu)/\Gamma(k\mu+1)$. Considering the largest term only, $\ln n_k \sim -k \ln k$. Thus, the slope of log-log plot of n_k versus k is relatively insensitive to μ [see Figs. 1(b), 3(b), and 3(d)].

Figure 3 shows the node distributions n_k under the noisy situations. Solid lines denote numerical solutions obtained from Eqs. (5), (4), and (3), while the circles and the triangles denote solutions from our simulations. As shown in Fig. 3, we observe that a random noise tends to hinder (accelerate) the emergence of highly linked nodes when $\alpha>0$ $(\alpha\leqslant 0)$. In order to verify the numerical solutions, we also plot $\langle n_k\rangle$ obtained from our simulations, which shows, for the most part, a good agreement with the solutions. This result can be also understood by the fact that for $\alpha>0$, $\widetilde{A}_k$ is always larger than A_k, but the difference become smaller as k increases. Thus, nodes with a small number of links tend to attract new nodes more strongly under the noisy condition, which results in relatively high probability n_k in the small k region [see Figs. 3(a) and 3(c)]. However, the situation is very different when $\alpha<0$. $\widetilde{A}_k$ is smaller than A_k when $k\leqslant\Delta$, while $\widetilde{A}_k$ is bigger than A_k when $k>\Delta$. Thus, highly linked nodes tend to attract new nodes more [see Figs. 3(b) and 3(d)].

IV. STABILITY OF NETWORKS

We now investigate the effects of random noise on the stability of a network against attacks. After we construct net-

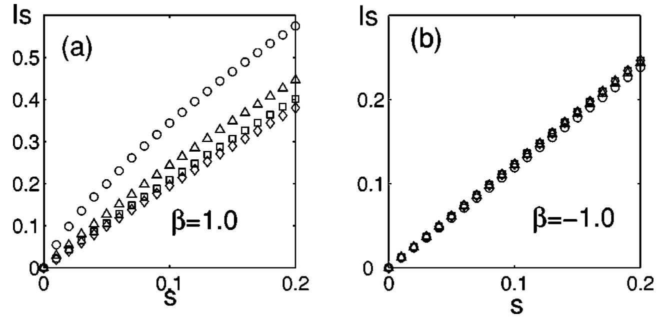

FIG. 4. Effect of noise on the stability of networks. The ratio of isolated nodes to the total number of nodes Is is plotted as a function of the fraction of deleted nodes s. (a) shows cases of attacking networks with a preferred deletion $B_k=k^\beta$ ($\beta=1.0$), while (b) shows cases of attacking networks with $\beta=-1.0$. The circles, the triangles, the squares, and the diamonds denote networks constructed with different strengths of random noise $\Delta=0$, 5, 10, and 20, respectively. (Note that the total number of nodes of networks before attacks is 10^5.)

works using our model, we *intentionally* remove nodes one by one. In order to select nodes to delete, we introduce a preferred deletion $B_k=k^\beta$. We assume that the probability of deleting a node with k links is proportional to B_k. In particular, when $\beta>0$, a highly linked node is more likely deleted, while, when $\beta<0$, a lesser linked node is deleted with higher probability [9]. In addition, when $\beta=0$, we randomly choose a node and delete it. Since the networks have an intrinsic random noise, we should define a noise-induced individual preferred deletion $\widetilde{B}_k^{(i)}$ in the same way we defined the individual attachment. For simplicity, we assume that a noise affects the counting of the number of links of individual nodes only. That is,

$$\widetilde{B}_k^{(i)}=\begin{cases}(\max[0,k+\tau^{(i)}])^\beta & (\beta>0)\\ \text{const} & (\beta=0)\\ (\max[1,k+\tau^{(i)}])^\beta & (\beta<0),\end{cases} \tag{6}$$

where $\tau^{(i)}$ is an uniformly distributed random integer ($\tau^{(i)}=-\Lambda,-\Lambda+1,\ldots,\Lambda$ and Λ is a positive integer) and β is a small real number.

In order to quantify the stability of a network, we count the number of isolated nodes in the network as we remove a node at each time step. Another way to quantify the stability of a network is to measure the *diameter* of the network suggested by Albert, Jeong, and Barabási [10]. However, if nodes are *sparsely* connected, a network tends to be fragmented when we remove nodes. In this case, we often observe sudden jumps in the diameter. Thus, counting the number of isolated nodes is a more proper method to determine the stability of the sparsely connected networks.

Figure 4 shows an effect of noise on the stability of networks. We construct networks using the model described in Sec. II fixing $\alpha=1.0$ and $\Delta=0$, 5, 10, and 20. After the total number of nodes in the networks becomes 10^5, we start to attack the networks. We choose a node at a time with a noise-induced preferred deletion $\widetilde{B}_k^{(i)}$ and remove the node

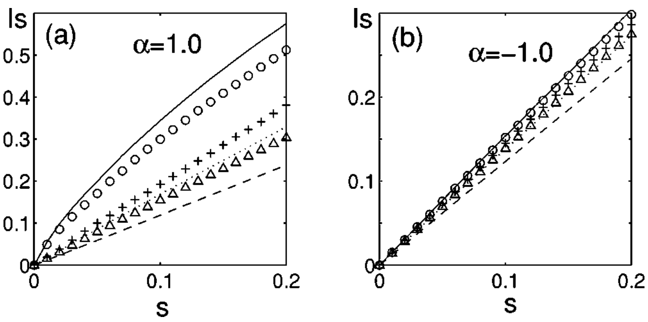

FIG. 5. Is versus s with different Λ. (a) shows cases of attacking a network constructed using a preferred attachment $A_k = k^{\alpha}$ ($\alpha = 1.0$), while (b) shows cases of attacking a network constructed with $\alpha = -1.0$. Solid lines ($\beta = 1.0$), dotted lines ($\beta = 0$), and dashed lines ($\beta = -1.0$) denote attacking a network without a noise, while the circles ($\beta = 1.0$), the pluses ($\beta = 0$), and the triangles ($\beta = -1.0$) denote attacking a network with a noise [(a) $\Lambda = 5$ and (b) $\Lambda = 2$]. (Note that the total number of nodes of networks before attacks is 10^5.)

and all links connected to it. [In order to simply illustrate the effect of noise, we do not consider any noise in the deletion process of nodes ($\Lambda = 0$). However, we will present the effect of noise on the deletion process at the end of this section.] We plot the changes in the ratio of isolated nodes to the total number of nodes Is in the networks as a function of the fraction of deleted nodes s under two different attacks. [See Fig. 4(a) for the attack with $\beta = 1.0$ and Fig. 4(b) for the attack with $\beta = -1.0$.] The circles, the triangles, the squares, and the diamonds denote networks constructed with different strengths of random noise, $\Delta = 0,5,10$, and 20, respectively.

As shown in Fig. 4, when $\beta > 0$, the number of isolated nodes becomes smaller in networks with strong noise than that in networks with weak noise. However, when $\beta < 0$, we observe different results. The number of isolated nodes is not much sensitive to the noise in the networks and even slightly increases as the noise in the network becomes stronger. This can be explained by the fact that the sturcture of networks is determined by the strength of noise [in particular, see Figs. 1, 3(a), and 3(c)]. When $\alpha > 0$, the number of highly linked nodes become smaller as Δ becomes larger. Thus, the network with strong noise is more invulnerable when attacked with $\beta > 0$, while it is relatively easily segmented when $\beta < 0$.

We consider the effect of noise in the deletion. We choose $\alpha = \pm 1$ and $\Delta = 0$, and add 10^5 nodes. After that we start to attack the networks with three different deletions ($\beta = -1.0,0,1.0$). Figure 5 shows the changes in the ratio of isolated nodes to the total number of nodes Is as a function of the fraction of deleted nodes s. Lines denote the ratio of isolated nodes without a noise ($\Lambda = 0$), while symbols denote with noise [$\Lambda = 5$ for Fig. 5(a) and $\Lambda = 2$ for Fig. 5(b)]. We observe that a network with a negative α is more stable than a network with a positive α in a sense that it has less isolated nodes when it is attacked with a positive β [see solid lines in Figs. 5(a) and 5(b)]. However, when the networks are attacked with a negative β we observe a relatively same result [see dashed lines in Fig. 5]. Since only a few nodes have very large number of links, while most of the nodes have a small number of links in the network constructed with a positive α, attacking the highly linked nodes results in a rapid increase of the number of isolated nodes, while attacking the lesser linked nodes results in a slow increase. However, for a network with a negative α, the link distribution of nodes are quite uniform, attacking either the highly linked nodes or the lesser linked nodes does not show much differences.

As shown in Fig. 5, the noise tends to stabilize (destabilize) the networks by reducing (increasing) the number of the isolated nodes when the networks are attacked with a positive (negative) β [see the circles ($\beta = 1.0$) and the triangles ($\beta = -1.0$) in Fig. 5]. We can define the average preferred deletion $\tilde{B}_k$ (B_k) from the individual deletion $\tilde{B}_k^{(i)}$ ($B_k^{(i)}$) in the same way as we defined $\tilde{A}_k$ (A_k) from $\tilde{A}_k^{(i)}$ ($A_k^{(i)}$). Then, we observe that $\tilde{B}_k \geqslant B_k$ and the difference becomes larger as k decrease under the noisy condition. It causes the probability of selecting nodes with a small number of links to be relatively higher when we attack the network with a positive β. Thus, the network under the noisy condition becomes more stable than the network without a noise (see solid lines and the circles in Fig. 5). However, when β is negative, $\tilde{B}_k$ is smaller than B_k in the small k region ($k \leqslant \Lambda$). Thus, the probability of selecting highly linked nodes becomes higher due to noise, which results in destabilizing the network under the noise situation (see the dashed lines and the triangles in Fig. 5).

V. CONCLUSION

We have presented the effects of random noise on network systems. We construct a growing network model, introducing noise-induced attachment $\tilde{A}_k$. A noise is observed to hinder (accelerate) the emergence of the highly linked nodes in a network with a positive (negative) α. Analytic and numerical solutions are presented as well and compared with our numerical simulations. In addition, we have also investigated the effects of noise on the stability of networks. After we construct a network using the model, we select a node with a probability which is proportional to the noise-induced deletion $\tilde{B}_k^{(i)}$ and delete it at each time step. The stability of a network is quantified by counting the number of isolated nodes as a function of the number of deleted nodes. We observe that noise stabilizes (destabilizes) a network when the network is attacked with a positive (negative) β. Although there is some tendency that noise in the construction of the network generally has some stabilizing effect, a proper choice of the deletion strategy can cancel this effect. We conjecture that our observations on the effects of a random noise are key features of many growing networks.

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[8] In many growing networks, it is often observed that a popular node becomes more and more popular. This phenomenon can be understood by the model with a positive power attachment $A_k \sim k^\alpha$ $(\alpha > 0)$. In particular, a linear attachment model was proposed by Barabási and Albert in Ref. [5]. However, negative power attachment may be relevant in some situations, such as computer terminals and server networks. People tend to connect to less occupied servers if other conditions are the same.
[9] It is a realistic assumption to attack a highly linked node with higher probability $(\beta > 0)$ in many networks. However, we may also observe a situation where nodes with a few links disappear easily, while highly linked nodes exist long $(\beta < 0)$. As a simple example, we can consider web pages in World Wide Web. Popular pages exist for long time, while unpopular pages easily disappear.
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